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The action with the Grassmann-odd Lagrangian for the supersymmetric classical Witten mechanics is constructed. It is shown that the exterior differential can be used for the connection between Grassmann-even and Grassmann-odd formulations of the classical dynamics in the superspace in both Hamilton's and Lagrange's approaches.

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1. There is a well-known close inter-relation between Hamilton's and Lagrange's formulations of the classical dynamics in the superspace when both of them are performed by means of the even (with respect to the Grassmann grading) attributes: the even Poisson bracket and the even Hamiltonian for the first formulation, and the action with the even Lagrangian for the last one. On the other hand, it was shown that Hamilton's dynamics for the systems, having an equal number of pairs of even and odd canonical variables, has equivalent Grassmann-odd Hamilton's formulation with the odd both Poisson bracket and Hamiltonian [1, 2]. Therefore, by analogy with the even case it was naturally assumed [2] that Grassmann-odd Lagrange's formulation of the dynamics for such systems has also to exist.

In this note the idea about the dynamics formulation based on the action with the Grassmann-odd Lagrangian is realized on the example of $d = 1, N = 2$ supersymmetric Witten's mechanics [3] in its classical version [4, 1]. Note, that the validity of the statement made in [2] concerning the existence of the dynamics formulation by means of the odd Lagrangian was also confirmed in [5].

2. Let us consider a system invariant with respect to the $N = 2$ $(\alpha = 1, 2)$ supersymmetry of the proper time t

$$t' = t + i\epsilon^\alpha\theta^\alpha \ , \qquad \theta'^\alpha = \theta^\alpha + \epsilon^\alpha \ .$$

By using the covariant derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\delta_{\alpha\beta}\theta^\beta\frac{\partial}{\partial t} \ ,$$

and two real scalar superfields

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$$\Phi(t,\theta^1,\theta^2) = q(t) + i\psi_\alpha(t)\theta^\alpha + iF(t)\theta^\alpha\theta_\alpha \ ,$$
$$\Psi(t,\theta^1,\theta^2) = \eta(t) + ia_\alpha(t)\theta^\alpha + i\Xi(t)\theta^\alpha\theta_\alpha \ ,$$

having the opposite values of the Grassmann grading g ($g(\Phi)=0, g(\Psi)=1$), the following supersymmetric action $\bar{S}$ with the Grassmann-odd Lagrangian $\bar{L}$ ($g(\bar{L})=1$) can be constructed

$$\bar{S} = \int dt d\theta_2 d\theta_1 \left[-\frac{1}{2} D^\alpha \Psi D_\alpha \Phi + i\Psi W(\Phi) \right] = \int dt \bar{L} \ , \tag{1}$$

where $W(\Phi)$ is an arbitrary real function of Φ and the index α is raised $D^\alpha = \epsilon^{\alpha\beta} D_\beta$ and lowered $\theta_\alpha = \epsilon_{\alpha\beta}\theta^\beta$ by means of the antisymmetric tensors $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$ ($\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^\alpha_\gamma$, $\epsilon^{12}=1$). Excluding after integration in (1) over the Grassmann variables θ^1, θ^2 the auxiliary fields F and Ξ with the help of their equations of motion, we obtain

$$\bar{L} = \dot{\eta}\dot{q} + \frac{i}{2}(a^\alpha \dot{\psi}^\alpha - \dot{a}^\alpha \psi^\alpha - 2a^\alpha\psi_\alpha W') - \eta \ (WW' + \frac{i}{2}\psi^\alpha\psi_\alpha W'') \ , \tag{2}$$

where the dot and the prime mean the derivatives with respect to t and q correspondingly. The odd Lagrangian $\bar{L}$ leads to the momenta

$$p = \frac{\partial \bar{L}}{\partial \dot{\eta}} = \dot{q} \ ; \qquad \pi = \frac{\partial \bar{L}}{\partial \dot{q}} = \dot{\eta} \ ; \tag{3a}$$

$$\pi^\alpha = \frac{\partial \bar{L}}{\partial \dot{a}^\alpha} = -\frac{i}{2}\psi^\alpha \ ; \qquad p^\alpha = \frac{\partial \bar{L}}{\partial \dot{\psi}^\alpha} = \frac{i}{2}a^\alpha \ , \tag{3b}$$

canonically conjugate to the coordinates η, q, a^α and ψ^α in the odd bracket

$$\{\eta, p\}_1 = \{q, \pi\}_1 = 1; \qquad \{a^\alpha, \pi^\beta\}_1 = \{\psi^\alpha, p^\beta\}_1 = \delta^{\alpha\beta} \ , \tag{4}$$

where the remaining odd-bracket relations between the canonical variables with zero in their right-hand sides are unwritten here. Relations (3a) express the velocities $\dot{q}$ and $\dot{\eta}$ in terms of the corresponding momenta, while (3b) define the constraints

$$\varphi^\alpha = \pi^\alpha + \frac{i}{2}\psi^\alpha \ ; \qquad f^\alpha = -p^\alpha + \frac{i}{2}a^\alpha \ , \tag{5}$$

which, as can be verified by using relations (4), are of the second class

$$\{\varphi^\alpha, f^\beta\}_1 = -i\delta^{\alpha\beta}; \qquad \{\varphi^\alpha, \varphi^\beta\}_1 = \{f^\alpha, f^\beta\}_1 = 0 \ . \tag{6}$$

We designate the Grassmann-even coordinates, momenta and constraints with the Latin letters, while the odd ones with the Greek letters. If we introduce the variables

$$\chi^\alpha = \pi^\alpha - \frac{i}{2}\psi^\alpha \ ; \qquad g^\alpha = -p^\alpha - \frac{i}{2}a^\alpha \ ,$$

which are canonically conjugate each other in the odd bracket (4)

$$\{\chi^\alpha, g^\beta\}_1 = i\delta^{\alpha\beta}; \qquad \{\chi^\alpha, \chi^\beta\}_1 = \{g^\alpha, g^\beta\}_1 = 0 \ ,$$

and whose odd-bracket relations with the constraints φ^α, f^α are vanished, then Dirac's odd bracket from any functions A and B take the form

$$\{A,B\}_1^{D.B.} = \{A,B\}_1 + i\{A,\varphi^\alpha\}_1\{f^\alpha,B\}_1 - i\{A,f^\alpha\}_1\{\varphi^\alpha,B\}_1 =$$
$$= A\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_\pi - \overleftarrow{\partial}_\pi\overrightarrow{\partial}_q + \overleftarrow{\partial}_\eta\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_\eta + i\overleftarrow{\partial}_{\chi^\alpha}\overrightarrow{\partial}_{g^\alpha} - i\overleftarrow{\partial}_{g^\alpha}\overrightarrow{\partial}_{\chi^\alpha}\right)B\ , \tag{7}$$

where $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ are right and left derivatives, and the notation $\partial_z = \frac{\partial}{\partial z}$ is introduced. Following from the Lagrangian (2) the total odd Hamiltonian, if subjected to the second-class constraints $\varphi^\alpha = 0$ and $f^\alpha = 0$, takes the form

$$\bar{H} = p\pi + \eta\ (WW' + \frac{i}{2}\chi_\alpha\chi^\alpha W'') + ig_\alpha\chi^\alpha W' \tag{8}$$

and with the use of Dirac's bracket (7) gives Hamilton's equations

$$\dot{x}^a = \{x^a, \bar{H}\}_1^{D.B.}$$

for the independent phase variables $x^a = (q, p, \chi^\alpha, \eta, \pi, g^\alpha)$

$$\dot{q} = p\ , \qquad \dot{p} = -\ WW' - \frac{i}{2}\chi_\alpha\chi^\alpha W''\ , \qquad \dot{\chi}^\alpha = \chi_\alpha W'\ , \tag{9a}$$

$$\dot{\eta} = \pi\ , \qquad \dot{\pi} = -\left\{\frac{\eta}{2}\ [(W^2)'' + \chi_\alpha\chi^\alpha W'''] + ig_\alpha\chi^\alpha W''\right\}\ , \qquad \dot{g}^\alpha = g_\alpha W' + \eta\chi_\alpha W''\ . \tag{9b}$$

Equations (9a) are Hamilton's equations for Witten's supersymmetric mechanics [3] in its classical version [1] which can be derived by means of Dirac's even bracket

$$\{A,B\}_0^{D.B.} = \{A,B\}_0 - i\{A,\varphi^\alpha\}_0\{\varphi^\alpha,B\}_0 = A\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q + i\overleftarrow{\partial}_{\chi^\alpha}\overrightarrow{\partial}_{\chi^\alpha}\right)B \tag{10}$$

with the help of the even Hamiltonian H

$$H = \frac{p^2 + W^2(q)}{2} + \frac{i}{2}\chi_\alpha\chi^\alpha W'(q)\ , \tag{11}$$

which both follow from the $N = 2$ supersymmetric action with the Grassmann-even Lagrangian L ($g(L) = 0$) (see, for example, [4])

$$S = \frac{1}{4}\int dt d\theta_2 d\theta_1\,[D^\alpha\Phi D_\alpha\Phi + 2iV(\Phi)] = \int dt L\ , \tag{12}$$

where $V'(\Phi) = 2W(\Phi)$ and the even Lagrangian after exclusion of the auxiliary field F is

$$L = \frac{1}{2}\ [\dot{q}^2 + i(\psi^\alpha\ \dot{\psi}^\alpha + \psi_\alpha\psi^\alpha W') - W^2]\ . \tag{13}$$

The momenta canonically conjugate to the coordinates q and ψ^α in the even bracket

$$\{q,p\}_0 = 1; \qquad \{\psi^\alpha, \pi^\beta\}_0 = -\delta^{\alpha\beta}\ ,$$

following from the even Lagrangian (13), have the form

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q}\ ; \qquad \pi^\alpha = \frac{\partial L}{\partial \dot{\psi}^\alpha} = -\frac{i}{2}\psi^\alpha\ .$$

The last relations define the second-class constraints

$$\varphi^\alpha = \pi^\alpha + \frac{i}{2}\psi^\alpha\ ; \qquad \{\varphi^\alpha, \varphi^\beta\}_0 = -i\delta^{\alpha\beta}\ , \tag{14}$$

which commute in the even bracket with the variables

$$\chi^\alpha = \pi^\alpha - \frac{i}{2}\psi^\alpha\ ; \qquad \{\chi^\alpha, \chi^\beta\}_0 = i\delta^{\alpha\beta}\ ,$$

entering into the definitions for the even Dirac bracket (10) and the even Hamiltonian (11). The even Hamiltonian (11) follows with the use of the second-class constraints restriction $\varphi^\alpha = 0$ from the total even Hamiltonian corresponding to the Lagrangian (13).

Equation (9b) can be obtained by taking the exterior differential d of the Hamilton equations (9a) for the Witten mechanics and performing the map λ:

$$\begin{gathered} dq \to \eta; \qquad dp \to \pi; \qquad d\chi^\alpha \to g^\alpha \\ d\psi^\alpha \to a^\alpha; \qquad d\pi^\alpha \to -p^\alpha; \qquad dF \to \Xi; \qquad d\varphi^\alpha \to f^\alpha\ . \end{gathered} \tag{15}$$

Note, that we identify the grading of the exterior differential d with the Grassmann grading g of the quantities θ^α ($g(d) = g(\theta^\alpha) = 1$), i.e., $g(dx^a) = g(x^a) + 1$. The composition $\lambda \circ d$ of the maps λ and d renders the even Hamiltonian (11) into the odd one (8)

$$dH \stackrel{\lambda}{\to} \bar{H}$$

3. The inter-relation of the brackets (10), (7) and of the corresponding to them Hamiltonians (11), (8) can be described by the following scheme. Hamilton's equations of a system expressed by means of the usual even Poisson-Martin bracket with the use of the Grassmann-even Hamiltonian H

$$\dot{x}^a = \{x^a, H\}_0 = \omega^{ab}\frac{\partial H}{\partial x^b}\ , \tag{16}$$

can be rewritten as

$$\dot{x}^a = \omega^{ab}\frac{\partial H}{\partial x^b} \equiv \omega^{ab}\frac{\partial(dH)}{\partial(dx^b)} \stackrel{\text{def}}{=} \{x^a, dH\}_1\ . \tag{17}$$

The exterior differential of the Hamilton equations (16) can be expressed in the form

$$d\dot{x}^a = (d\omega^{ab})\frac{\partial(dH)}{\partial(dx^b)} + (-1)^{g(a)+g(b)}\omega^{ab} d(\frac{\partial H}{\partial x^b})\ . \tag{18}$$

If x^a and dx^a are considered as independent variables, then the order of the exterior and partial differentiations can be changed in the second term in the right-hand side of relation (18) which, thus, take the form

$$d\dot{x}^a = (d\omega^{ab})\frac{\partial(dH)}{\partial(dx^b)} + (-1)^{g(a)}\omega^{ab}\frac{\partial(dH)}{\partial x^b} \stackrel{\text{def}}{=} \{dx^a, dH\}_1\ .$$

Thus, we introduced by definition the odd bracket

$$\{A,B\}_1 =$$
$$= A\left[\frac{\overleftarrow{\partial}}{\partial x^a}\,\omega^{ab}\,\frac{\overrightarrow{\partial}}{\partial(dx^b)} + (-1)^{g(a)}\frac{\overleftarrow{\partial}}{\partial(dx^a)}\,\omega^{ab}\,\frac{\overrightarrow{\partial}}{\partial x^b} + \frac{\overleftarrow{\partial}}{\partial(dx^a)}\,(d\omega^{ab})\,\frac{\overrightarrow{\partial}}{\partial(dx^b)}\right]B\ , \tag{19}$$

that reproduces with the use of the odd Hamiltonian dH the Hamilton equations (17) for the phase coordinates x^a, obtained by means of the even bracket with the help of the even Hamiltonian H, and gives, besides, the equations for dx^a. Note, that the last term in the definition (19) disappears if we use a canonical form for the even bracket (16). As can be verified with the use of the properties of the matrix ω^{ab} entering in the even bracket (16), the bracket (19) does possess the properties necessary for the odd bracket. In connection with a similar scheme see also the paper [6].

It is interesting to note that, by using this scheme, the Grassmann-odd Hamilton formulation for the supersymmetric one-dimensional oscillator fulfilled by means of the odd bracket

$$\dot{x^a} = \{x^a, \bar{H}\}_1 = x^a\left(\overleftarrow{\partial}_{\theta^1}\overrightarrow{\partial}_q - \overleftarrow{\partial}_q\overrightarrow{\partial}_{\theta^1} + \overleftarrow{\partial}_{\theta^2}\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_{\theta^2}\right)\bar{H}$$

with the odd Hamiltonian $\bar{H} = q\theta^2 - p\theta^1$ can be obtained from the even Hamilton formulations of both the Bose-oscillator and the Fermi-oscillator which are described correspondingly in the Poisson and Martin even brackets

$$\{A,B\}_0^{P.B.} = A\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_p\right)B; \qquad \{A,B\}_0^{M.B.} = -iA\left(\overleftarrow{\partial}_{\theta^1}\overrightarrow{\partial}_{\theta^1} + \overleftarrow{\partial}_{\theta^2}\overrightarrow{\partial}_{\theta^2}\right)B$$

with the even Hamiltonians $H = (p^2+q^2)/2$ and $\tilde{H} = i\theta^1\theta^2$, respectively. Under this, the operation of taking the exterior differential must be accompanied by the designations $dq = \theta^2$, $dp = -\ \theta^1$ in the Bose-oscillator case and $id\theta^1 = q$, $id\theta^2 = p$ for the Fermi-oscillator.

In general case if we have a Hamilton system which dynamics is described by means of the bracket $\{A,B\}_\epsilon$ with the help of the Hamiltonian $\overset{\epsilon}{H}$

$$\dot{x^a} = \{x^a, \overset{\epsilon}{H}\}_\epsilon = \overset{\epsilon}{\omega}{}^{ab}\,\frac{\partial \overset{\epsilon}{H}}{\partial x^b}\ , \tag{20}$$

where ϵ ($\epsilon = 0,1$) is the Grassmann parity of both the bracket and the Hamiltonian, then the Hamilton equations for the phase coordinates x^a and the equations for their differentials dx^a, obtaining by a differentiation of equations (20), can be reproduced by the following bracket of the opposite Grassmann parity

$$\{A,B\}_{\epsilon+1} =$$
$$= A\left[\frac{\overleftarrow{\partial}}{\partial x^a}\overset{\epsilon}{\omega}{}^{ab}\frac{\overrightarrow{\partial}}{\partial(dx^b)} + (-1)^{g(a)+\epsilon}\frac{\overleftarrow{\partial}}{\partial(dx^a)}\overset{\epsilon}{\omega}{}^{ab}\frac{\overrightarrow{\partial}}{\partial x^b} + \frac{\overleftarrow{\partial}}{\partial(dx^a)}\,(d\,\overset{\epsilon}{\omega}{}^{ab})\,\frac{\overrightarrow{\partial}}{\partial(dx^b)}\right]B \tag{21}$$

with the help of the Hamiltonian $d\overset{\epsilon}{H}$ ($g(d\overset{\epsilon}{H}) = \epsilon+1$), that is,

$$\dot{x^a} = \{x^a, \overset{\epsilon}{H}\}_\epsilon = \ \{x^a, d\overset{\epsilon}{H}\}_{\epsilon+1}\ ; \qquad \dot{dx^a} = d(\{x^a, \overset{\epsilon}{H}\}_\epsilon) = \ \{dx^a, d\overset{\epsilon}{H}\}_{\epsilon+1}\ .$$

We can again verify, by using the properties of the matrix $\overset{\epsilon}{\omega}{}^{ab}$ for the bracket (20), that (21), in fact, satisfies all the properties necessary for the bracket $\{A,B\}_{\epsilon+1}$.

As in the case of Hamilton's formulations there is an interconnection between Lagrange's equations corresponding to the Lagrangians of the different Grassmann parities L (13) and $\bar{L}$ (2), because the odd Lagrangian $\bar{L}$ (2) is related by means of the redefinition λ (15) with the exterior differential dL of the even Lagrangian (13). Indeed, Lagrange's equations corresponding to the Lagrangian $\overset{\epsilon}{L}(q^a,\dot{q}^a)$ with the Grassmann parity ϵ can be written in the two equivalent forms

$$\frac{d}{dt}\left(\frac{\partial\overset{\epsilon}{L}}{\partial\dot{q}^a}\right)-\frac{\partial\overset{\epsilon}{L}}{\partial q^a}=0\Longleftrightarrow\frac{d}{dt}\left(\frac{\partial(d\overset{\epsilon}{L})}{\partial(d\dot{q}^a)}\right)-\frac{\partial(d\overset{\epsilon}{L})}{\partial(dq^a)}=0\ , \tag{22a, b}$$

while the equations obtained by taking the differential of (22a) have the form

$$\frac{d}{dt}\left(\frac{\partial(d\overset{\epsilon}{L})}{\partial\dot{q}^a}\right)-\frac{\partial(d\overset{\epsilon}{L})}{\partial q^a}=0\ . \tag{22c}$$

Equations (22b) together with (22c) can be considered as Lagrange's equations for the system described in the configuration space with the coordinates q^a, dq^a by the Lagrangian $d\overset{\epsilon}{L}$ of the Grassmann parity $\epsilon+1$.

Note also that if the Lagrangian $\overset{\epsilon}{L}$ has the constraints $\varphi^i(q^a,p_a=\partial\overset{\epsilon}{L}/\partial\dot{q}^a)$ satisfying the relations in the bracket corresponding to $\overset{\epsilon}{L}$

$$\{\varphi^i,\varphi^k\}_\epsilon=f^{ik}\ ,$$

then the Lagrangian $d\overset{\epsilon}{L}$ will possess the constraints $\varphi^i(q^a,p_a=\partial(d\overset{\epsilon}{L})/\partial(d\dot{q}^a))$, coinciding with those following from $\overset{\epsilon}{L}$, and $d\varphi^i$ obeying the relations

$$\{\varphi^i,d\varphi^k\}_{\epsilon+1}=f^{ik};\qquad\{\varphi^i,\varphi^k\}_{\epsilon+1}=0\ ,$$

which follow from the related with $d\overset{\epsilon}{L}$ bracket expression (21) that in the case is without the last term in their right-hand side (cf., e.g., equations (14) with (5), (6)).

4. Thus, it is shown that for the given formulation of the dynamics (either in Hamilton's or in Lagrange's approach) with the equations of motion for the dynamical variables z^a we can construct, by using the exterior differential, such a formulation, having the opposite Grassmann parity, that reproduces the former equations for z^a and gives, besides, the equations for their differentials dz^a.

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